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Small Magnetofluid-Dynamic Peristaltic Motions Inside an Annular Circular Cylindrical Induction Compressor

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The small magnetofluid-dynamic peristaltic motions inside an annular circular cylindrical induction compressor are studied. The compressor consists of a circular cylindrical traveling wave tube of radius r_0 on which is impressed a purely sinusoidal current sheet of the form $nI\ \exp i(kz - \omega t)\hat{\theta}$, where nI is the number of ampere turns per unit length, k the wave number, and ω the circular frequency. Inside the tube, an annulus $r_0 \geq r \geq r_1$ is filled with a highly conducting fluid that is constrained at the ends not to move in the axial direction. The fluid to be pumped moves in an annulus $r_1 \geq r \geq r_2$ and is separated from the conducting fluid by an impermeable flexible diaphragm. The electromagnetically induced motions of these two inviscid and incompressible fluids, when the electromechanical coupling is weak, i.e., in the limit of small magnetic Reynolds number (based on wave speed and wavelength), are examined analytically. From the nature of the axial component of the body force induced in the constrained conducting fluid, it is shown that a time-averaged constant axial pressure gradient is induced which is transmitted to the pumped fluid by virtue of the mechanical coupling between them. If now, the induced, purely oscillatory, radial force component is sufficient to pinch and trap the pumped fluid, the ensuing motion of the latter would consist of trapped packets of fluid traveling in the wave direction with the wave speed of the diaphragm (equal to the speed of the traveling current sheet) against the pressure gradient induced in it.

I. Introduction

RECENTLY, a great deal of interest has been shown in the application of the principles of magnetofluid-dynamics to the compression and acceleration of poorly conducting liquids (e.g., sea water), with an eye toward its possible application to the propulsion of undersea craft.[1, 2] To date, most of all of the schemes proposed involve the direct interaction of the magnetic and electric fields with the poor conductor. Since the electrical conductivity of sea water is approximately six orders of magnitude less than ordinary liquid metal conductors, such schemes are doomed to failure because the magnitudes of the induced currents are so small that tremendous field strengths are required to produce significant forces. However, this low conductivity limitation could be circumvented, for example, if a pumping scheme could be devised wherein the electromagnetic field, instead of acting directly on the poor conductor, is made to act directly on an intermediate working fluid of large conductivity. If now, the working fluid and the fluid pumped are mechanically coupled, the large forces induced in the working fluid could conceivably be transmitted to the pumped fluid.

A possible scheme that utilizes the foregoing concept is shown schematically in Fig. 1. An annulus $r_0 \geq r \geq r_1$, inside a circular cylindrical tube of radius r_0, is filled with a highly conducting fluid that is constrained at the ends not to move in the axial direction. The pumped fluid, moving in the annulus $r_2 \leq r \leq r_1$, is separated from the conducting fluid by a flexible impermeable diaphragm. Impressed on a transmission line, in the form of a coil wound around the cylinder, is a purely sinusoidal traveling current sheet of the form $nI\ \exp i(kz - \omega t)\hat{\boldsymbol{\theta}}$, where nI is the amplitude of the number of ampere turns per unit length, k the wave number, ω the circular frequency, and $\omega/k = V$, the wave speed. If the relative speed between the fluid conductor and wave is different from zero, then from the circular symmetry, closed azimuthal currents will be induced in the conductor. These currents, when crossed with the radial and axial components of the traveling **B** field associated with the traveling current sheet, produce axial and radial body forces, respectively. Since the conducting fluid is constrained at the ends not to move in the axial direction, the axial velocity of the fluid

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conductor averaged across its annulus per cycle is zero, so that an axial pressure gradient is induced in the conductor given by

$$\sigma V B_r^2 = \partial p/\partial z \tag{1}$$

where B_r is the radial component of the traveling **B** field and p is the fluid pressure. This pressure gradient, which is never negative because of the quadratic dependence on the radial magnetic field component, is transmitted to the pumped fluid via the mechanical coupling between the fluids. If now the radial forces (which will be shown to be purely oscillatory) in the conductor are sufficient to pinch and trap the pumped fluid, the ensuing motion of the pumped fluid would consist of trapped packets of fluid traveling with the wave speed of the diaphragm (equal to the speed of the traveling current sheet) against the pressure gradient induced in it by virtue of the axial body forces induced in the conducting liquid. In effect, this peristaltic action could be compared to a series of compressors, each giving a finite pressure rise to the pumped fluid, each wavelength corresponding to a single compressor stage.

The twofold purpose of this paper is as follows: 1) to investigate analytically the nature of the electromechanically induced motions inside the compressor, both of the fluid conductor and the pumped fluid, when the electromechanical coupling is weak, i.e., in the limit of small magnetic Reynolds number (based on wave speed and wavelength), and 2) to examine, in this limiting case of small motions, under what conditions and to what extent the forementioned conjectured behavior of the compressor is predicted analytically.

II. Vector Potential and Induced Forces

In order to carry out the magnetofluid-mechanical calculation, we must first find the appropriate expressions for the induced forces in the fluid conductor. Let us assume that the pumped fluid has zero conductivity and free space permeability μ_0, so that, as far as the electromagnetics is concerned, the region between $r_1 \geq r \geq 0$, where r_1 is the equilibrium position of the diaphragm, is considered a homogeneous vacuum. We denote this region as region 1. The annular region filled with conducting fluid $r_0 \geq r \geq r_1$ is denoted as region 2, and the region outside the tube $\infty \geq r \geq r_0$ as region 3. Starting with Maxwell's equations, neglecting the displacement current, introducing the vector potential, and following the deirvation of Ref. 3, the following partial differential equations and boundary conditions are obtained for the corresponding azimuthal component of the vector potential[3]:

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{1}{r}\frac{\partial A_1}{\partial r} - \frac{A_1}{r^2} + \frac{\partial^2 A_1}{\partial z^2} = 0 \qquad r_1 \geq r \geq 0 \tag{2}$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} + \frac{\partial^2 A_2}{\partial z^2} = \mu_0\sigma_2 \frac{\partial A_2}{\partial t} \qquad r_0 \geq r \geq r_1 \tag{3}$$

$$\frac{\partial^2 A_3}{\partial r^2} + \frac{1}{r}\frac{\partial A_3}{\partial r} - \frac{A_3}{r^2} + \frac{\partial^2 A_3}{\partial z^2} = 0 \qquad \infty \geq r \geq r_0 \tag{4}$$

at $r = r_0$,

$$\partial A_2/\partial z = \partial A_3/\partial z \tag{5}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_2) - \frac{1}{r}\frac{\partial}{\partial r}(rA_3) = \mu_0 nI \exp i(kz - \omega t) \tag{6}$$

at $r = r_1$,

$$\partial A_1/\partial z = \partial A_2/\partial z \tag{7}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_2) - \frac{1}{r}\frac{\partial}{\partial r}(rA_1) = 0 \tag{8}$$

with the requirements that

$$A_1 \text{ be finite at } r = 0 \tag{9}$$

$$A_3 \to 0 \text{ as } r \to \infty \tag{10}$$

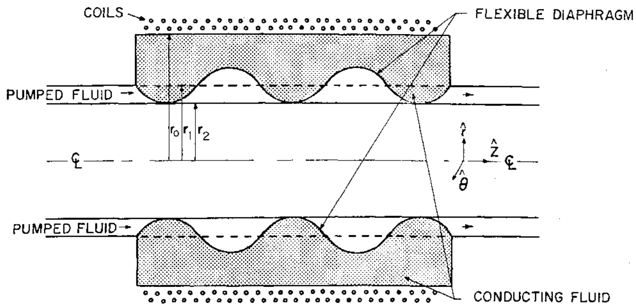

Fig. 1 Schematic diagram of peristaltic annular circular cylindrical induction compressor.

In Eq. (3), σ_2 represents the conductivity of the fluid conductor, and unlike the corresponding equation of Ref. 3, we have set the zero-order velocity of the constrained conducting fluid equal to zero.

The solution of the three region system (2–10) is given in Ref. 3. In particular, we obtain the following solution for the azimuthal component of the vector potential in region 2:

$$A_2(r, z, t) = R_e\left[\mu_0 nI \frac{M_2K_1(kr_0)I_1(\alpha r) + M_4K_1(kr_0)K_1(\alpha r)}{M_2M_3 + M_4M_5} \times \exp[i(kz - \omega t)]\right] \tag{11}$$

where

$$\begin{aligned} M_2 &= K_1(\alpha r_1)I_0(kr_1) - \alpha I_1(kr_1)K_0(\alpha r_1) \\ M_3 &= \alpha I_0(\alpha r_0)K_1(kr_0) - kI_1(\alpha r_0)K_0(kr_0) \\ M_4 &= \alpha I_1(kr_1)I_0(\alpha r_1) - I_1(\alpha r_1)I_0(kr_1) \\ M_5 &= \alpha K_0(\alpha r_0)K_1(kr_0) - kK_1(\alpha r_0)K_0(kr_0) \end{aligned} \tag{12}$$

In (11) and (12), I_0 and I_1 represent the modified Bessel functions of the first kind of order zero and one, respectively, whereas K_0 and K_1 represent the corresponding Bessel functions of the second kind. The symbol α is defined by

$$\alpha^2 = k^2(1 - i\beta) \tag{13}$$

where $\beta = \mu_0\sigma_2\omega/k^2$ and represents the fluid magnetic Reynolds number based on wave speed and wavelength. Expanding (11) about small β, and using the identity

$$I_0(kr_0)K_1(kr_0) - I_1(kr_0)K_0(kr_0) = -(1/kr_0)$$

we obtain, to the zeroth order in β,

$$A_2(r,z,t) = -\mu_0 nIr_0K_1(kr_0)I_1(kr)\cos(kz - \omega t) \tag{14}$$

From the definitions of the azimuthal component of the current density and radial and axial components of the magnetic field in terms of the vector potential,[3] we have

$$\left.\begin{aligned} j_\theta &= -\sigma_2(\partial A_2/\partial t) = \\ &\qquad \sigma_2\omega\mu_0 nIr_0K_1(kr_0)I_1(kr)\sin(kz - \omega t) \\ B_r &= -(\partial A_2/\partial z) = \\ &\qquad -k\mu_0 nIr_0K_1(kr_0)I_1(kr)\sin(kz - \omega t) \\ B_z &= (1/r)(\partial/\partial r)(rA_2) = \\ &\qquad -k\mu_0 nIr_oK_1(kr_0)I_0(kr)\cos(kz - \omega t) \end{aligned}\right\} \tag{15}$$

Thus, the radial and axial components of the induced body force acting on the fluid conductor, to order β, are given by

$$F_r = (\mathbf{j} \times \mathbf{B})_r = j_\theta B_z = \tfrac{1}{2}\beta\mu_0 n^2I^2k^3r_0^2K_1^2(kr_0)I_1(kr)I_0(kr)\sin 2(kz - \omega t) \tag{16}$$

$$F_z = (\mathbf{j} \times \mathbf{B})_z = -j_\theta B_r = \tfrac{1}{2}\beta\mu_0 n^2I^2k^3r_0^2K_1^2(kr_0)I_1^2(kr)[1 - \cos 2(kz - \omega t)] \tag{17}$$

It is to be noted from (16) and (17) that the radial force component is purely sinusoidal, propagating at twice the frequency of the traveling current sheet, whereas the axial component (which is never negative) consists of the sum of a nonpropagating part plus a propagating component, at twice the input frequency, which is 90° out of phase with the radial vibration.

III. Fluid Flow Equations and Boundary Conditions

We treat the fluid flow problem under the following five assumptions: 1) the motion is axisymmetric, 2) both fluids are inviscid, 3) no azimuthal component of velocity (this is reasonable since the assumption that $\mathbf{j}$ is purely azimuthal implies that no azimuthal forces are induced in the conducting fluid; thus if the conducting fluid has zero azimuthal velocity initially, it will have zero azimuthal velocity for all time thereafter), 4) both fluids are incompressible and, in order to avoid gravitational effects, are assumed to be of the same density ρ, and 5) the electromagnetic force in the poorly conducting pumped fluid is taken to be zero. Let the subscripts c and w refer to the fluid mechanical variables of the fluid conductor and the pumped fluid, respectively.

Letting $\mathbf{q}_c = \beta(v_c\hat{\mathbf{r}} + u_c\hat{\mathbf{z}})$ and $P_c = P_{0c} + \beta p_c$, where βv_c and βu_c are the perturbed radial and axial velocity components, respectively, P_{0c} the undisturbed pressure, and βp_c the perturbation pressure, the continuity and momentum equations, to order β, in the fluid conductor $r_0 \geq r \geq r_1$, are

$$\frac{\partial}{\partial r}(rv_c) + \frac{\partial}{\partial z}(ru_c) = 0 \tag{18}$$

$$\rho\frac{\partial v_c}{\partial t} = -\frac{\partial p_c}{\partial r} + \frac{1}{\beta}F_r \tag{19}$$

$$\rho\frac{\partial u_c}{\partial t} = -\frac{\partial p_c}{\partial z} + \frac{1}{\beta}F_z \tag{20}$$

where F_r and F_z are given by (16) and (17), respectively. Again, letting $\mathbf{q}_w = U_0\hat{\mathbf{z}} + \beta(v_w\hat{\mathbf{r}} + u_w\hat{\mathbf{z}})$ and $P_w = P_{0w} + \beta p_w$, where U_0 is the unperturbed uniform axial velocity of the pumped fluid through its annulus $r_2 \leq r \leq r_1$, the continuity and momentum equations, to order β, in the pumped fluid are

$$\frac{\partial}{\partial r}(rv_w) + \frac{\partial}{\partial z}(ru_w) = 0 \tag{21}$$

$$\rho\left(\frac{\partial v_w}{\partial t} + U_0\frac{\partial v_w}{\partial z}\right) = -\frac{\partial p_w}{\partial r} \tag{22}$$

$$\rho\left(\frac{\partial u_w}{\partial t} + U_0\frac{\partial u_w}{\partial z}\right) = -\frac{\partial p_w}{\partial z} \tag{23}$$

The boundary conditions that the system (18–23) must satisfy are as follows. At the rigid walls $r = r_0$ and $r = r_2$, we require

$$v_c(r_0, z, t) = 0 \tag{24}$$

$$v_w(r_2, z, t) = 0 \tag{25}$$

The following considerations lead to the boundary conditions that must be satisfied at the equilibrium position of the diaphragm $r = r_1$. The diaphragm, representing a material surface, requires that the component of fluid velocity normal to the instantaneous position of the diaphragm surface in each of the fluid regions be equal to the velocity of the diaphragm in its normal direction. Since a small perturbation solution is being generated, we assume that the amplitude of the diaphragm is very small compared to its wavelength and that its instantaneous slope is of order β. In this case, the fluid velocities normal to the diaphragm surface is given by v_c and v_w. Thus, we have the boundary condition

$$v_c(r_1, z, t) = v_w(r_1, z, t) \tag{26}$$

The second boundary condition requires that the total pressure (magnetic plus fluid mechanical) be continuous across the unstressed diaphragm. Since the magnetic field components are continuous across the diaphragm (no current sheet), the magnetic pressures are automatically equal. Hence, we require

$$P_{0c} = P_{0w} \qquad p_c(r_1, z, t) = p_w(r_1, z, t) \tag{27}$$

One final boundary condition must be considered, i.e., that imposed by the ends of the device. We assume that at the ends of the device the conducting fluid is bounded by walls such that $u_c = 0$ at the ends. Away from the ends there is therefore allowed no time-average mass flow of conducting fluid across any position z. Mathematically,

$$\int_{r_1}^{r_0} \langle\rho u_c\rangle_t\, 2\pi r dr = 0 \tag{28}$$

We now seek a solution of the system (18–23) subject to the boundary conditions (24–28).

IV. Solution

A. Fluid Conductor Region

Consider the fluid conductor region first. Multiplying (19) and (20) by r, differentiating (19) with respect to r and (20) with respect to z, adding, and dividing by r, we obtain, using continuity (18),

$$0 = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_c}{\partial r}\right) - \frac{\partial^2 p_c}{\partial z^2} + \frac{1}{\beta}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{\partial F_z}{\partial z}\right\}$$

Using the definitions of F_r and F_z given by (16) and (17) and carrying out the differentiations indicated by the right-hand side, we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_c}{\partial r}\right) + \frac{\partial^2 p_c}{\partial z^2} = \frac{1}{2}\mu_0 n^2 I^2 k^4 r_0^2 K_1^2(kr_0) \times [I_1^2(kr) - I_0^2(kr)]\sin 2(kz - \omega t) \tag{29}$$

It is readily verified by direct substitution that

$$\bar{P}_c = \tfrac{1}{4}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) I_0^2(kr)\sin 2(kz - \omega t)$$

is the particular integral of Eq. (29). We take as our general solution

$$p_c = \tfrac{1}{4}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) I_0^2(kr)\sin 2(kz - \omega t) + [C_1 I_0(2kr) + C_2 K_0(2kr)]\sin 2(kz - \omega t) + (C_3 + C_4\ln r)z + f_1(t) \tag{30}$$

where it is again easily verified that each of the terms in the second row satisfies the homogeneous equation and where the unknown function $f_1(t)$ and constants C_1, C_2, C_3, and C_4 remain to be determined.

Substituting (16) and (30) into (19) yields

$$\rho(\partial v_c/\partial t) = -\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) I_1(kr) I_0(kr)\sin 2((kz - \omega t) - 2k[C_1 I_1(2kr) + C_2 K_1(2kr)]\sin 2(kz - \omega t) - C_4 z/r$$

Integrating yields

$$\rho v_c = -(1/2\omega)\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) I_1(kr) I_0(kr)\cos 2(kz - \omega t) - k/\omega[C_1 I_1(2kr) + C_2 K_1(2kr)]\cos 2(kz - \omega t) - C_4 zt/r$$

The boundary condition (24) requires $v_c(r_0, z, t) = 0$ for all z and t, hence $C_4 = 0$ and

$$C_1 I_1(2kr_0) + C_2 K_1(2kr_0) = -\tfrac{1}{2}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) I_1(kr_0) I_0(kr_0) \tag{31}$$

Substituting (17) and (30) into (20) yields

$$\rho(\partial u_c/\partial t) = \tfrac{1}{2}\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) I_1^2(kr) - \tfrac{1}{2}\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0)[I_1^2(kr) + I_0^2(kr)] \cos 2(kz - \omega t) - [C_1 I_0(2kr) + C_2 K_0(2kr)]2k \cos 2(kz - \omega t) - C_3$$

Integrating yields

$$\rho u_c = \tfrac{1}{2}\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) I_1^2(kr) t + (1/4\omega)\mu_0 n^2 I^2 k^3 r_0^2 \times K_1^2(kr_0)[I_1^2(kr) + I_0^2(kr)] \sin 2(kz - \omega t) + k/\omega[C_1 I_0(2kr) + C_2 K_0(2kr)] \sin 2(kz - \omega t) - C_3 t \quad (32)$$

We have then, at this point, u_c given by (32), v_c and p_c given by

$$\rho v_c = -(1/2\omega)\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) I_1(kr) I_0(kr) \cos 2(kz - \omega t) - k/\omega[C_1 I_1(2kr) + C_2 K_1(2kr)] \cos 2(kz - \omega t) \quad (33)$$

$$p_c = \tfrac{1}{4}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) I_0^2(kr) \sin 2(kz - \omega t) + [C_1 I_0(2kr) + C_2 K_0(2kr)] \sin 2(kz - \omega t) + C_3 z + f_1(t) \quad (34)$$

and the relation (31) coming from the boundary condition (24).

B. Pumped Fluid Region

Multiplying (22) and (23) by r, differentiating (22) with respect to r and (23) with respect to z, adding, and dividing by r, gives, using continuity (21),

$$\frac{1}{r}\frac{\partial}{\partial_r}\left(r\frac{\partial p_w}{\partial r}\right) + \frac{\partial^2 p_w}{\partial z^2} = 0 \quad (35)$$

We take as general solution of (35)

$$p_w = [C_5 I_0(2kr) + C_6 K_0(2kr)] \sin 2(kz - \omega t) + (C_7 + C_8 \ln r)z + f_2(t) \quad (36)$$

Substituting (36) into (22) gives

$$\rho\left(\frac{\partial v_w}{\partial t} + U_0\frac{\partial v_w}{\partial z}\right) = -2k[C_5 I_1(2kr) + C_6 K_1(2kr)] \sin 2(kz - \omega t) - C_8\frac{z}{r}$$

Integrating yields

$$\rho v_w = \frac{-k}{(\omega - U_0 k)}[C_5 I_1(2kr) + C_6 K_1(2kr)] \times \cos 2(kz - \omega t) - \frac{C_8}{r}\frac{z^2}{2U_0}$$

The boundary condition (25), requiring $v_w(r_2, z, t) = 0$ for all z and t, gives $C_8 = 0$ and

$$C_5 I_1(2kr_2) + C_6 K_1(2kr_2) = 0 \quad (37)$$

Substituting (36) into (23) gives

$$\rho\left(\frac{\partial u_w}{\partial t} + U_0\frac{\partial u_w}{\partial z}\right) = -2k[C_5 I_0(2kr) + C_6 K_0(2kr)] \sin 2(kz - \omega t) - C_7$$

Integrating yields

$$\rho u_w = \frac{k}{(\omega - U_0 k)}[C_5 I_0(2kr) + C_6 K_0(2kr)] \times \sin 2(kz - \omega t) - C_7 t \quad (38)$$

We have then, at this point, u_w given by (38), v_w and p_w given by

$$\rho v_w = \frac{-k}{(\omega - U_0 k)}[C_5 I_1(2kr) + C_6 K_1(2kr)] \times \cos 2(kz - \omega t) \quad (39)$$

$$p_w = [C_5 I_0(2kr) + C_6 K_0(2kr)] \sin 2(kz - \omega t) + C_7 z + f_2(t) \quad (40)$$

and the relation (37) coming from the boundary condition (25).

C. Solution (continued)

The boundary condition (26), using (33) and (39), gives

$$C_1 I_1(2kr_1) + C_2 K_1(2kr_1) - C_5\frac{\omega}{(\omega - U_0 k)} I_1(2kr_1) - C_6\frac{\omega}{(\omega - U_0 k)} K_1(2kr_1) = -\frac{1}{2}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) I_1(kr_1) I_0(kr_1) \quad (41)$$

The boundary condition (27), using (34) and (40), gives

$$C_3 = C_7 \qquad f_1(t) = f_2(t) \quad (42)$$

$$C_1 I_0(2kr_1) + C_2 K_0(2kr_1) - C_5 I_0(2kr_1) - C_6 K_0(2kr_1) = -\tfrac{1}{4}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) I_0^2(kr_1) \quad (43)$$

Equations (31, 37, 41, and 43) form a system of four inhomogeneous linear algebraic equations for the four unknown constants C_1, C_2, C_5, and C_6. Inspection of the inhomogeneous terms shows that each of these constants is proportional to $\frac{1}{2}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0)$. Thus, let

$$C_1 = \tfrac{1}{2}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) C_{IC}$$
$$C_2 = \tfrac{1}{2}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) C_{KC}$$
$$C_5 = \tfrac{1}{2}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) C_{IW}$$
$$C_6 = \tfrac{1}{2}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0) C_{KW}$$

Hence, from (32–34 and 38–40) we obtain

$$\rho u_c = \tfrac{1}{2}\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) I_1^2(kr) t - C_3 t + (1/2\omega)\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0)\{\tfrac{1}{2}[I_1^2(kr) + I_0^2(kr)] + C_{IC} I_0(2kr) + C_{KC} K_0(2kr)\}\cdot \sin 2(kz - \omega t) \quad (44)$$

$$\rho v_c = -(1/2\omega)\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0)\{I_1(kr) I_0(kr) + C_{IC} I_1(2kr) + C_{KC} K_1(2kr)\}\cdot \cos 2(kz - \omega t) \quad (45)$$

$$p_c = \tfrac{1}{2}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0)\{\tfrac{1}{2} I_0^2(kr) + C_{IC} I_0(2kr) + C_{KC} K_0(2kr)\} \sin 2(kz - \omega t) + C_3 z + f_1(t) \quad (46)$$

$$\rho u_w = \frac{1}{2(\omega - U_0 k)}\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) \times \{C_{IW} I_0(2kr) + C_{KW} K_0(2kr)\} \sin 2(kz - \omega t) - C_3 t \quad (47)$$

$$\rho v_w = -\frac{1}{2(\omega - U_0 k)}\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) \times \{C_{IW} I_1(2kr) + C_{KW} K_1(2kr)\} \cos 2(kz - \omega t) \quad (48)$$

$$p_w = \tfrac{1}{2}\mu_0 n^2 I^2 k^2 r_0^2 K_1^2(kr_0)\{C_{IW} I_0(2kr) + C_{KW} K_0(2kr)\} \sin 2(kz - \omega t) + C_3 z + f_1(t) \quad (49)$$

where we have still to evaluate C_3 and $f_1(t)$.

The constant C_3 is determined from the boundary condition (28). Substituting (44) into (28), and remembering that the average of the periodic term over a cycle is zero, we obtain

$$t\int_{r_1}^{r_0}\{\tfrac{1}{2}\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) I_1^2(kr) - C_3\} 2\pi r dr = 0$$

Thus,

$$C_3 \tfrac{1}{2}(r_0^2 - r_1^2) = \tfrac{1}{2}\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0)\int_{r_1}^{r_0} r I_1^2(kr) dr = \tfrac{1}{2}\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0)\{\tfrac{1}{2}r^2[I_1^2(kr) - I_0(kr) I_2(kr)]\}_{r_1}{}^{r_0}$$

or

$$C_3 = \frac{\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0)}{2(r_0^2 - r_1^2)}\{r_0^2[I_1^2(kr_0) - I_0(kr_0) I_2(kr_0)] - r_1^2[I_1^2(kr_1) - I_0(kr_1) I_2(kr_1)]\} \quad (50)$$

The unknown function $f_1(t)$ is afforded the same interpretation as is given to the integration function in the usual derivation of the nonsteady Bernoulli equation, that is, $f_1(t)$ is determined from the time dependence of the perturbed input pressure of the incoming fluid being pumped.

Finally, the motion of the diaphragm is obtained as follows. Let βr_d be the radial displacement of the diaphragm to order β, then

$$\partial r_d/\partial t = v_c(r_1, z, t)$$

Integrating (45) with respect to t, we obtain

$$r_d(z, t) = (1/4\omega^2\rho)\mu_0 n^2 I^2 k^3 r_0^2 K_1^2(kr_0) \cdot \{I_1(kr_1)I_0(kr_1) + C_{IC}I_1(2kr_1) + C_{KC}K_1(2kr_1)\} \sin 2(kz - \omega t) \quad (51)$$

V. Discussion

The resulting fluid mechanical motions inside the annular traveling wave peristaltic compressor, in the weak coupling limit, are given by Eqs. (44–49) with the motion of the tensionless diaphragm separating the two fluids given by Eq. (51). Apart from the particular radial dependence generated by the solution, the following behavior for the different fluid mechanical variables in each of the fluid regions is observed:

1) The axial velocity component in each region consists of the sum of a linear time-dependent term superimposed on a purely sinusoidal traveling component of the same wave speed but of twice the frequency of the input current sheet. The generation of the linear time-dependent term is due naturally to the existence of the nonperiodic part in the expression for the axial body force. The solution generated, then, is both a small time and small coupling solution and is invalid for large times. In any actual case, however, the magnitude of the axial velocity component cannot increase indefinitely with time, being limited eventually both by viscous effects and the fact that when the velocity reaches the wave speed coupling ceases, the electromagnetic force goes to zero, and the fluid coasts thereafter.

2) The radial velocity component in each region consists of a purely sinusoidal traveling component only, which is 90° out of phase with the corresponding periodic component of the axial velocity. It should be noted that the singularity $U_0 = \omega/k$ in Eqs. (47) and (48) is only apparent. Examination of the linear set of equations from which the constants C_{IW} and C_{KW} are derived shows that each constant is proportional to $(\omega - U_0 k)$.

3) As conjectured in the Introduction, the diaphragm motion is indeed characterized by the axial propagation of a purely sinusoidal wave down the diaphragm whose speed is equal to that of the traveling current sheet but at twice the frequency, and whose amplitude (and therefore pinching and trapping ability) is proportional to $\mu_0(nI)^2$, i.e., the magnetic pressure associated with the current sheet.

4) The pressure in each region consists, apart from the integration functions $f_1(t) = f_2(t)$, of a purely sinusoidal traveling part plus a term linear in z. The existence, in the pumped fluid, of this latter term is the crux of the entire analysis and makes its appearance only when the axial motion in the conducting fluid is constrained. If the motion in the conductor were not constrained, this term would never appear and the device would behave like an accelerator rather than a pump. As an accelerator, the axial body force would simply go into developing a purely sinusoidal pressure gradient plus increasing the axial velocity of the working fluid. However, as conjectured in the Introduction, as a pump, the work that ordinarily would be done by the axial body force in accelerating the conducting fluid now goes into the work done in pumping the incoming fluid against the pressure gradient induced in it by virtue of the mechanical coupling between the two fluids. Further, one would imagine that the constant pressure gradient induced in the pumped fluid would be proportional to the time-averaged axial body force evaluated at r_1, i.e., at the equilibrium position of the diaphragm. Examination of C_3, however, shows that the pressure gradient is equal to the time-averaged axial body force averaged over the conductor annulus. Since the magnitude of the body force decreases inward from the coils, the analysis predicts a pumping force larger than suspected.

In conclusion, although one cannot, in general, predict the behavior of a nonlinear system from the periodic behavior of the corresponding linear one, still, it may well be that the compressor behavior, when the electromechanical coupling is strong, corresponds to some limit cycle situation whose gross features (apart from the linear time-dependent terms) are similar to these just obtained. In any case, the proof of the pudding lies in experimental verification.

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